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On time-dependent AdS/CFT

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Abstract: We clarify aspects of the holographic AdS/CFT correspondence that are typical of lorentzian signature, to lay the foundation for a treatment of time-dependent gravity and conformal field theory phenomena. We provide a derivation of bulk-to-boundary propagators associated to advanced, retarded and Feynman bulk propagators, and provide a better understanding of the boundary conditions satisfied by the bulk fields at the horizon. We interpret the subleading behavior of the wavefunctions in terms of specific vacuum expectation values, and compute two-point functions in our framework. We connect our bulk methods to the closed time path formalism in the boundary field theory.

Keywords: Conformal Field Models in String Theory, AdS-CFT and dS-CFT Correspondence.

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1. Introduction

The AdS/CFT correspondence [1] originated in part from the comparison of absorption amplitudes calculated using IIB supergravity, and from the worldvolume action of $D3$-branes [2] in lorentzian signature (see e.g. [3] for a brief review). A clear euclidean computational prescription for the duality was formulated in [4, 5]. Since then, the AdS/CFT correspondence has been mostly taken into a euclidean setting, with great success.

Nevertheless, it has been stressed many times that important physical problems, that may have a natural resolution in a holographic framework are time-dependent. If we want to study processes in gravity such as the formation or evaporation of a black hole, and we want to obtain a holographic description of the process in terms of a dual conformal field theory, then time-dependence will surely enter the game. Clearly, it will not be an easy task to obtain such a description.

Time-dependent quantum field theory is a difficult subject by itself, independent of the complications associated to holography (see [6, 7, 8, 9] and references thereto). Indeed, like in a classical wave problem, we can (instead of only asking questions about the overlap of asymptotic states) try to evolve a quantum field theoretical system according to the wave equation of the quantum field theory at hand, naturally using retarded and advanced propagators (instead of Feynman propagators only). These problems are often harder to solve than the problems in scattering theory although recently good progress has been made thanks to numerical methods. One might hope that a better understanding of time-dependent AdS/CFT can shed some light on time-dependent problems in strongly coupled (supersymmetric conformal) field theories at large N.

In this paper we take one more step towards a holographic dictionary between Lorentzian AdS and a time-dependent conformal quantum field theory on the boundary. We make a connection between previous works on lorentzian AdS/CFT, and we tie together the prescriptions given in the literature [4, 10]–[14], in an elementary approach that we hope clarifies the basic issues. We will assume throughout that we work in a string theoretic context, where the AdS/CFT correspondence has been tested most convincingly.

We start in section 2 by briefly reviewing the wave function for a scalar field in AdS_{d+1}, and as an extra, we give an intuitive explanation of the fact that boundary conditions on the scalar field are needed for low values of the mass. In section 3 we review the standard quantisation of a scalar field in Poincaré coordinates. By reinterpreting the bulk Green functions in section 4, we are able to relate the regularisation prescription in Minkowksi space to a radial regularisation prescription. Thus, we link causal properties in the boundary to radial boundary conditions in the bulk. In section 6, we apply the formalism to compute two-point functions and to interpret the subleading behavior of bulk solutions to the equations of motion. We assembled conclusions and remarks in section 7.

2. Wave functions on the Poincaré patch

In this section, we clarify some assumptions that underlie the standard quantisation of scalar fields in AdS_{d+1}. That will clear the ground for quantising scalar fields on AdS_{d+1} and deriving their Green functions in the next section.

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2.1 A rough analysis

The metric in Poincaré coordinates (u, x^μ) is $(u \in [0, \infty))$ (see appendix A for our conventions):

$$ds^2 = u^{-2}(dx^\mu)^2 + \frac{du^2}{u^2}\,, \tag{2.1}$$

and we will also frequently make use of the radial coordinate $r = 1/u$. To get a first idea of the spectrum for a scalar field, we briefly discuss solutions to the wave-equation:

$$(\Box - m^2)\Phi(r, x^\mu) = 0\,. \tag{2.2}$$

Using separation of variables, we can write a generic solution as a linear combination of factorised wave functions $\Phi(r, x^\mu) = e^{-iEt}e^{ip_j x^j}R(r)$. We distinguish two types of solutions, depending on the causal nature of the Minkowski momentum p^μ. When the Minkowski momentum is spacelike ($p^2 > 0$), we have:

$$\Phi = c e^{-iEt} e^{ip_j x^j} r^{-\frac{d}{2}} K_\nu\left(\sqrt{p^2}\frac{1}{r}\right). \tag{2.3}$$

The solution is regular in the interior (near $r = 0$), and (delta-function) normalizable, for $0 \leq \nu = \sqrt{m^2 + d^2/4} < 1$. We assume that the Breitenlohner-Freedman [15, 16] bound $m^2 \geq -d^2/4$ is satisfied[1]. For timelike momentum ($p^2 < 0$) the two solutions regular in the interior are:

$$\Phi^\pm = c_\pm e^{-iEt} e^{ip_j x^j} r^{-\frac{d}{2}} J_{\pm\nu}\left(\sqrt{-p^2}\frac{1}{r}\right) \tag{2.4}$$

(for ν not an integer). For $\nu \geq 1$ only Φ^+ is normalizable, while for $\nu < 1$ both modes are. The meaning of this fact will be clarified in the following subsections.

To obtain more insight in the radial behavior of these modes, and in particular, the way their normalisability properties depend on the value of ν, we introduce in the next subsection an auxiliary one-dimensional radial differential equation that will be of good use throughout our paper.

2.2 Boundary conditions

To clarify which eigenfunctions we should consider and what boundary conditions should be imposed, it is useful to reduce the scalar wave equation to a one-dimensional problem. In the (u, x^μ) Poincaré coordinate system, the factors of the function $\Phi(u, x^\mu) =$

[1] As an aside, we note that an excitation that violates the BF bound is the analogue of a tachyon in AdS. These excitations play a crucial role for strings on AdS_3, where in the twisted, winding sector, there are stable excitations built on the component of the particle Hilbert space that is tachyonic and associated to the radial momentum modes [17]. Note also that in the euclidean AdS_3 setting, the only available modes are exactly the ones that correspond to these tachyonic modes in the lorentzian AdS_3 (i.e. it is only the continuous representations of $Sl(2)$ that are present for $\mathrm{SL}(2,\mathbb{C})$ [18, 19]). It would be interesting to understand whether one can make sense of these tachyonic modes in QFT on AdS in general, in an interacting setting.

$\phi(x^\mu)u^{d/2-1/2}f(u)$ will satisfy the wave equation in Minkowski space with $(\text{mass})^2 = \lambda$ and the one-dimensional Sturm-Liouville equation in the radial direction u with eigenvalue λ [20]:

$$\begin{aligned}(\Box_M - \lambda)\phi(x^\mu) &= 0 \\ -f(u)'' + \frac{m^2 + d^2/4 - 1/4}{u^2} f(u) &= \lambda f(u)\,. \end{aligned} \tag{2.5}$$

More generally, we will find it useful to discuss the inhomogeneous Sturm-Liouville problem with source $f_s(u)$:

$$f(u)'' + (\lambda - q(u))f(u) = f_s(u)\,, \tag{2.6}$$

with parameter $\nu = +\sqrt{m^2 + d^2/4}$ and potential $q(u) = \frac{\nu^2 - 1/4}{u^2}$.

Necessity of boundary conditions. The radial equation for the scalar wavefunction can be written in terms of the differential operator $-\partial_u^2 + \frac{\nu^2-1/4}{u^2}$. Before we discuss this operator in detail, we consider the problem of wave solutions on the real half-line $[0,\infty[$ to gain some intuition on when it is necessary to impose boundary conditions on the wavefunction at $u = 0$.

First of all, if we study the problem without potential ($q(u) = 0$), then, if we don't specify a boundary condition at the end of the half-line ($u = 0$) for the plane wave (which is thought of as an eigenfunction of $-\partial_t^2 + \partial_u^2$), we don't have a well-defined physical problem, and that is reflected in the fact that the operator $-\partial_u^2$ is not self-adjoint on the space of functions without specific boundary condition. We need to specify a boundary condition at zero, because the wave reaches zero easily, and we need to know how it bounces back to determine its full evolution. It turns out that the specification of a boundary condition is in one-to-one correspondence to the specification of a self-adjoint extension of the operator ∂_u^2 on the half-line [21].

In general, whether we need to specify a boundary condition at zero, in a Schrödinger problem on the half-line, depends on how fast the potential grows at zero. If it grows very fast, it takes care of the boundary condition automatically. The only normalisable solution will have a fixed phase as it bounces off the wall at $u = 0$. If the potential does not grow fast enough, we need to specify the boundary condition that will pick one out of two normalisable solutions to the wave equation.

How fast is fast enough? The critical behavior of the potential is c/u^2. It turns out [21] that when $c \geq 3/4$, the potential grows fast enough to have only one normalisable solution. When $c < 3/4$, we need to specify a boundary condition. Since $\nu^2 = c + 1/4$, we find that when $0 \leq \nu < 1$, we will need to specify a boundary condition at zero for the Schrödinger problem on the half line. When $\nu \geq 1$ we don't need to specify a boundary condition, and there will be only one normalisable solution.

That explains the pattern we observed above for the normalisability of the scalar wave functions in AdS_{d+1}. It also shows that we need to impose an extra boundary condition in the case $0 \leq \nu < 1$. The choice of boundary condition makes the propagation of fields on the AdS space well-defined for that mass range. If we do not specify the boundary condition,

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we don't know how the scalar wave bounces back off the boundary of the spacetime. Thus, we conclude from this analysis that we can just consider the eigenfunctions proportional to J_ν when $\nu \geq 1$. However, when $0 \leq \nu < 1$, we need to specify boundary conditions. That will single out a linear combination of $J_{\pm\nu}$, and a single bound state wave function proportional to K_ν [20, 22]. With this extra understanding, we can systematically treat the quantisation of the AdS scalar field. Our analysis here can be read as an intuitive restating of facts laid bare in [15, 16, 20].

In this paper we concentrate on the case $\nu \geq 1$[2]. For $\nu < 1$ there are two types of conformal boundary conditions [23] that single out the $\Phi^{\pm}$ modes and that exclude a bound state solution. We hope to return to the renormalisation group flow physics associated to more general boundary conditions (see e.g. [23, 24]) and the interpretation of the bound state elsewhere. We thus have clarified a little the assumptions that underlie the ordinary quantisation of scalar fields in AdS_{d+1} and can now proceed with a clear conscience in standard fashion.

3. Bulk propagator

In this section we take a first look at the computation of the bulk Feynman, retarded and advanced propagators (see also [25, 26]). We concentrate on quantizing the modes $\Phi \propto J_\nu$ proportional to the Bessel function with positive index $\nu \geq 1$.

3.1 Quantisation

We write the general solution in terms of positive and negative frequency components, a normalisation constant $c(p)$ and annihilation operators $f(p)$:

$$\begin{aligned} \Phi = & \int_{E>\sqrt{p_j p^j}} dE \int d^{d-1}p_j f(p)c(p)e^{ip.x}r^{-\frac{d}{2}}J_\nu\left(\sqrt{-p^2}\frac{1}{r}\right) + \\ & + f(p)^\dagger c(p)^* e^{-ip.x}r^{-\frac{d}{2}}J_\nu\left(\sqrt{-p^2}\frac{1}{r}\right). \end{aligned} \tag{3.1}$$

To normalize the operators f, we compute the scalar product between the positive frequency eigenfunctions:

$$\begin{aligned} \left(\Phi^{(+)}(p), \Phi^{(+)}(p')\right) &= -i\int dr dx^j \sqrt{-g}g^{00}\Phi^{(+)}(p)^* \overleftrightarrow{\partial_0}\Phi^{(+)}(p') \\ &= (2\pi)^{d-1}\delta(p_j - p'_j)c(p)^*c(p')\int_0^\infty dr.r^{d-1}.r^{-2}.r^{-d}(E+E') \times \\ &\quad \times J_\nu\left(\sqrt{-p^2}\frac{1}{r}\right)J_\nu\left(\sqrt{-(p')^2}\frac{1}{r}\right). \qquad (3.2) \\ &= (2\pi)^{d-1}\delta(p_j - p'_j)c(E,p_j)^*c(E',p_j)(E+E') \times \\ &\quad \times \int_0^\infty du u J_\nu(\sqrt{-p^2}u)J_\nu(\sqrt{-(p')^2}u). \qquad (3.3) \end{aligned}$$

[2]We moreover will brush over important subtleties associated to integer ν.

We have for $\nu > 1$:

$$f(s) = \int_0^\infty s^{1/2} J_\nu(su) u \int_0^\infty t^{1/2} J_\nu(tu) f(t) du dt \tag{3.4}$$

from which it follows that:

$$\int_0^\infty duu J_\nu(su) J_\nu(tu) = s^{-1}\delta(s-t)\,. \tag{3.5}$$

Using this formula, we find that the inner product evaluates to:

$$(\Phi^{(+)}(E,p_j), \Phi^{(+)}(E',p'_j)) = |c(p)|^2 2(2\pi)^{d-1}\delta(p_j - p'_j)\delta(E-E')\,. \tag{3.6}$$

Thus we put $c(p) = \frac{1}{\sqrt{2}}(2\pi)^{-\frac{d-1}{2}}$. We then postulate:

$$[f(p), f^\dagger(p')] = \delta(p-p') \tag{3.7}$$

to quantise the fields. It can then be checked that the quantum fields satisfy the standard commutation relations recalled in appendix A.

3.2 Green functions

Consider first the Wightman function G^+, which we can rewrite in terms of the Wightman function G_M^+ in Minkowski space [20]:

$$\begin{aligned} G^+(x,x') &= \langle 0|\Phi(x)\Phi(x')|0\rangle \\ &= \int_{\sqrt{p_j p^j}}^\infty dE \int \frac{d^{d-1}p_j}{(2\pi)^{d-1}} e^{ip.(x-x')} r^{-\frac{d}{2}} r'^{-\frac{d}{2}} \frac{1}{2} J_\nu\left(\sqrt{-p^2}\frac{1}{r}\right) J_\nu\left(\sqrt{-p^2}\frac{1}{r'}\right) \\ &= \int_0^\infty d\lambda \int \frac{d^d p}{(2\pi)^{d-1}} e^{ip.(x-x')}\theta(E)\delta(p^2+\lambda) r^{-\frac{d}{2}} r'^{-\frac{d}{2}} \frac{1}{2} J_\nu\left(\sqrt{\lambda}\frac{1}{r}\right) J_\nu\left(\sqrt{\lambda}\frac{1}{r'}\right) \\ &= \int_0^\infty d\lambda G_M^+(x^\mu, x'^\mu; \sqrt{\lambda}) r^{-\frac{d}{2}} r'^{-\frac{d}{2}} \frac{1}{2} J_\nu\left(\sqrt{\lambda}\frac{1}{r}\right) J_\nu\left(\sqrt{\lambda}\frac{1}{r'}\right)\,. \end{aligned} \tag{3.8}$$

In fact, an analogous analysis makes clear that there is a similar relation between all Green functions for the Poincaré patch of AdS_{d+1} and the Green functions on M^d. Indeed, the Wightman functions $G^\pm$ for a scalar field in AdS_{d+1} and those in Minkowski space, $G_M^\pm$, are related as above, and the time-ordering (θ-functions) agree in the Poincaré coordinates and Minkowski space. Thus, all Green functions in Poincaré coordinates can be straightforwardly derived from the Green functions in Minkowski space.

Let us be a little more specific. We will be interested in computing the retarded, advanced and Feynman bulk Green functions $G_B \in \{G_R, G_A, -G_F\}$. We have:

$$G^\pm(x,x') = \int_0^\infty d\lambda \int \frac{d^{d-1}p}{(2\pi)^{(d-1)}} \int_{C^\pm} \frac{dE}{2\pi i}\frac{1}{p^2+\lambda} e^{ip.(x-x')} r^{-\frac{d}{2}} r'^{-\frac{d}{2}} \frac{1}{2} J_\nu\left(\sqrt{\lambda}\frac{1}{r}\right) J_\nu\left(\sqrt{\lambda}\frac{1}{r'}\right)\,. \tag{3.9}$$

where the contours $C^{\pm}$ coincide with the Minkowski contours $C^{\pm}$ encircling one of two poles, as discussed in appendix A. There we also listed how to obtain the standard contours C^B for the bulk propagators G_B which can be expressed as:

$$G_B(x,x') = \int_0^\infty d\lambda \int \frac{d^{d-1}p}{(2\pi)^{(d-1)}} \int_{C^B} \frac{dE}{2\pi} \frac{1}{p^2+\lambda} e^{ip\cdot(x-x')} \times \\ \times r^{-\frac{d}{2}} r'^{-\frac{d}{2}} \frac{1}{2} J_\nu\left(\sqrt{\lambda}\frac{1}{r}\right) J_\nu\left(\sqrt{\lambda}\frac{1}{r'}\right). \tag{3.10}$$

The contours C^B enumerated in the appendix correspond to the usual ϵ-prescriptions: for the Feynman propagator $G_F : p^2 \to p^2 - i\epsilon$, for the retarded one $G_R : p^2 \to p^2 - i\epsilon E$ and $G_A : p^2 \to p^2 + i\epsilon E$ for the advanced propagator. That defines the bulk propagators in AdS_{d+1}.

Up till now, the treatment has been fairly standard. We can compare these bulk propagators with the ones obtained in global coordinates, and we expect them to agree (up to time-ordering) on the grounds that the vacua in Poincaré coordinates and in global coordinates only differ in their vacuum energy, not in the definition of positive and negative frequency modes [25, 27]. We check their agreement in appendix B. In the next section, we will reinterpret the expression for the bulk propagator in our holographic context.

4. Reinterpreting the bulk Green function

For our purposes it will be important to be able to take a different perspective on the bulk Green function G_B. We have several ways of reading formula (3.10) for G_B. One is as the Fourier transform of the Green function for the radial problem (later denoted $g(u,u')$), where the radial Green function is written in the form of an integral over eigenfunctions. The second way is to read it as a Bessel transform of a Green function in Minkowski space. i.e. we can interpret the role of the factor $\frac{1}{p^2+\lambda}$ in two ways, either as associated to a Minkowski Green function, or as associated to a radial Green function. That also implies that, after implementing the ϵ-prescription, we can either associate it to the integral over the energy E (leading to standard Minkowski propagators), or as regularising our radial Green function. To understand the second perspective better, we collect some useful ingredients.

4.1 A relation of the bulk propagator to a 1d propagator

Suppose we have a bulk Green function G_B that satisfies:

$$(\Box_x - m^2) G_B(x,x') = -\frac{1}{\sqrt{-g}} \delta(x-x'). \tag{4.1}$$

When we naively write it as:

$$G_B = \frac{1}{(2\pi)^d} \int dE \int dp_j e^{-iE(t-t')} e^{ip_j(x^j - x'^j)} g(u,u'), \tag{4.2}$$

then it is easy to derive that $g(u,u')$ has to satisfy ($\lambda = E^2 - p_j p^j$):

$$\left(\lambda u^2 + u^{d+1}\partial_u(u^{-d+1}\partial_u) - m^2\right) g(u,u') = -u^{d+1}\delta(u-u'). \tag{4.3}$$

When we define a Green function for the 1d problem by:

$$\partial_u^2 k_\lambda(u,u') + \left(\frac{-m^2 + 1/4 - d^2/4}{u^2} + \lambda\right) k_\lambda(u,u') = -\delta(u,u') \tag{4.4}$$

then g is related to k_λ by:

$$g(u,u') = u^{\frac{d-1}{2}} u'^{\frac{d-1}{2}} k_\lambda(u,u'). \tag{4.5}$$

Thus, we obtain a bulk propagator G_B as the Fourier transform of a Green function for the radial problem (when we take into account some extra powers of the radial coordinate). That is a first, quick way to view the abovementioned change of perspective.

4.2 Radial bulk propagator

Next, we discuss how to obtain the radial bulk propagator $k_\lambda(u,u')$ on the interval $[u_0, \infty[$ (see section 4.10 of [22]). We introduced an infrared cut-off in the radial direction, $u_0 > 0$, which excises the region near the boundary of the AdS space at $u = 0$. We need to study the problem for imaginary values of λ, and in particular we will take λ to have (at least a small) positive imaginary part. (This is necessary to make the wavefunction near the radial origin $r = 0$, or $u = \infty$, strictly square integrable.) After some computation, we find that the radial bulk propagator with boundary condition $k(u, u_0) = 0$ is given by:

$$k_{\lambda=s^2}(u,u') = -\frac{\pi}{2H_\nu^{(1)}(u_0 s)} u^{1/2} H_\nu^{(1)}(us)(u')^{1/2}(J_\nu(u's)Y_\nu(u_0 s) - Y_\nu(u's)J_\nu(u_0 s))$$
$$\text{for } u \geq u', \tag{4.6}$$

and a similar expression with u and u' interchanged for $u < u'$. It can be checked that if we take a naive $u_0 \to 0$ limit in this expression, we recuperate the radial Green function for the Sturm-Liouville problem on the half-line [22], which is directly related to the radial part of the bulk Green function (3.10).

4.3 Radial bulk-to-boundary propagator

For later use, we discuss how to obtain a bulk-to-boundary propagator (in momentum space) from the bulk propagator for the one-dimensional problem. It is easy to see that the only eigenfunction that behaves nicely at $u = \infty$, that satisfies the inhomogeneous boundary condition $f(u_0) = f_0$ and that is associated to $\lambda = s^2$ (with real and imaginary part of s larger than zero) is given by:

$$f(u) = f_0 \frac{u^{1/2} H_\nu^{(1)}(us)}{u_0^{1/2} H_\nu^{(1)}(u_0 s)}. \tag{4.7}$$

How do we obtain this solution from the bulk propagator for this problem? In this regularised problem, we can easily derive from the radial propagator (4.6), for $u' \to u_0$:

$$\partial_{u'} k_\lambda(u,u') \to \frac{u^{1/2} H_\nu^{(1)}(us)}{(u_0)^{1/2} H_\nu^{(1)}(u_0 s)}, \tag{4.8}$$

yielding the correct (trivial) bulk-to-boundary Green function we found above, as expected from general techniques used in solving wave equations (see e.g. [28]).

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It is clear that a naive limit of the bulk Green function, where we consider $u_0 \to 0$, or more precisely $u_0 s \to 0$, yields the bulk Green function that we find in the problem on the real positive half-line [22]. Note though that we would neglect terms that could become of equal strength when $us \to 0$. That would cause problems in finding the correct bulk-to-boundary propagator in the non-regularised problem. Physically speaking, the high momentum modes on the brane ruin the limit (this is a UV phenomenon), or, in other words, the bulk long distance IR phenomena associated to the boundary at $u = 0$ are not properly taken into account and need to be IR regulated from the bulk supergravity point of view. This UV/IR correspondence is familiar by now [29].

Note that the sign of the imaginary part of $s = \sqrt{\lambda}$ was decisive in choosing between the Hankel functions $H^{(1,2)}$. Indeed, had we taken the case where λ has negative imaginary part, we would have obtained very similar results with $H^{(2)}$ replacing $H^{(1)}$. The ϵ-prescription, or imaginary part of the eigenvalue, determines the radial part of the wavefunction uniquely.

5. Deep throat

We have now assembled the ingredients to be able to derive the bulk-to-boundary propagators K_B associated to bulk propagators G_B, when we cut off the AdS_{d+1} space at finite radius $r_0 = 1/u_0$. We need to reinterpret our bulk propagator as the Fourier transform of a radial Green function, and take the appropriate regularisation prescription into account in the radial part. The Feynman bulk-to-boundary propagator for instance is the Fourier transform of the one-dimensional bulk-to-boundary propagator associated to the $p^2 \to p^2 - i\epsilon$ or $s^2 \to s^2 + i\epsilon$ prescription:

$$\begin{aligned} K_F(u,x;x') &= -\partial'_u G_F(u,u') \qquad \text{at} \quad u' = u_0 \\ &= \int \frac{d^d p}{(2\pi)^d} e^{ip\cdot(x-x')} \frac{u^{d/2} H_\nu^{(1)}(u\sqrt{-p^2+i\epsilon})}{u_0^{d/2} H_\nu^{(1)}(u_0\sqrt{-p^2+i\epsilon})}\,, \end{aligned} \tag{5.1}$$

where we reinstalled the appropriate power of the radial coordinate. From this, it is easy to see that $K_F(u_0) = \delta(x^\mu - x'^\mu)$, as expected. For the advanced and retarded bulk-to-boundary Green functions, we need to adapt our radial regularisation to the sign of the energy E. We obtain:

$$\begin{aligned} K_{R,A}(u,x;x') &= \partial'_u G_{R,A}(u,u') \quad \text{at} \quad u' = u_0 \\ &= \int \frac{d^d p}{(2\pi)^{(d)}} e^{ip\cdot(x-x')} \left(\theta(\pm E) \frac{u^{d/2} H_\nu^{(1)}(u\sqrt{-p^2+i\epsilon})}{u_0^{d/2} H_\nu^{(1)}(u_0\sqrt{-p^2+i\epsilon})} + \right. \\ &\qquad \left. + \,\theta(\mp E) \frac{u^{d/2} H_\nu^{(2)}(u\sqrt{-p^2-i\epsilon})}{u_0^{d/2} H_\nu^{(2)}(u_0\sqrt{-p^2-i\epsilon})} \right). \end{aligned} \tag{5.2}$$

In this case too, it is easy to see that $K_B(u_0) = \delta(x^\mu - x'^\mu)$.

It is clear that the Feynman bulk-to-boundary Green function is a specific analytic continuation of the euclidean bulk-to-boundary propagator as described in [4, 11]. The bulk-to-boundary propagator contains waves that move inward for positive energy modes, and outward for negative energy modes. (We consider incoming modes to be the ones that travel towards the horizon at $r = 0$.) We showed that the ϵ-prescription in the Minkowski part of the bulk propagator determines the radial part for the propagator.

Our derivation illuminates the presciption given in [14]. Indeed, for the retarded and advanced propagators, we *derived* the correct boundary conditions at the horizon at $u = \infty$. That is a crucial ingredient in making the AdS/CFT prescription work. We therefore not only reproduced the fact that indeed the retarded propagator is associated to incoming boundary conditions on the bulk scalar field [14], but we will also see that the contribution to the two-point function at the horizon automatically vanishes once the appropriate ϵ-prescription is taken into account. This puts two ingredients of the prescription in [14] on a firm footing.

6. Correlation functions

6.1 Boundary behavior

We reformulate the boundary value problem for AdS_{d+1} in order to clarify the structure of the general solution to the equations of motion. After this general discussion, we return to the specific conclusions we can draw for the particular choices of bulk-to-boundary propagators that we defined in the previous section.

From the equations of motion, it becomes clear that the most general (non-normalisable) solution behaves as (see e.g. [11, 24]):

$$\Phi(u, x^\mu) = u^{d-\Delta_+}(\phi_-(x^\mu) + O(u^2)) + u^{\Delta_+}(\phi_+(x^\mu) + O(u^2)) \tag{6.1}$$

where $\Delta_\pm = d/2 \pm \nu$ and $\nu \geq 0$. It is clear then that the first term dominates near the boundary at $u = 0$ and is associated to non-normalisable behavior near the boundary for $\nu \geq 1$. The second term correponds to leading normalisable boundary behavior.

We note that a generic solution with non-normalisable boundary behavior specified by ϕ_- can be obtained as follows: derive a bulk-to-boundary Green function K_B from the bulk propagator G_B, and write a solution to the equation of motion satisfying the boundary condition ϕ_-. If we do not add any other term to the solution, we have also fixed the subleading ϕ_+ behavior. Picking a specific bulk propagator G_B and, hence, K_B can thus be understood as associating a particular subleading behavior ϕ_+ to each leading behavior ϕ_- [11]. We will analyse and interpret their precise relation in the next section. A general solution can be obtained by adding a normalisable solution to the one obtained in the above prescription. Of course, adding a normalisable mode will influence the subleading ϕ_+ boundary behavior [11].

6.2 Subleading behavior

As a preliminary to studying the two-point functions, we analyse the subleading behavior of specific wave functions, associated to the bulk-to-boundary propagators that we derived.

First of all, let's study the bulk solution associated to the Feynman bulk-to-boundary propagator K_F:

$$\begin{aligned} K_F &= \int \frac{d^d p}{(2\pi)^d} e^{ip\cdot(x-x')} \frac{u^{d/2} H_\nu^{(1)}(u\sqrt{-p^2+i\epsilon})}{u_0^{d/2} H_\nu^{(1)}(u_0\sqrt{-p^2+i\epsilon})} \\ \Phi(u,x^\mu) &= \int dx^{\mu\prime} \phi_-(x^{\mu\prime}) K_F(x^\mu, x^{\mu\prime}; u)\,. \end{aligned} \tag{6.2}$$

Using the expansion of the Hankel functions for small arguments, and the Fourier transform of $(-p^2+i\epsilon)^\nu$ [30], it is possible to show that the wavefunction Φ behaves as:

$$\Phi(u,x^\mu) \simeq \left(\frac{u}{u_0}\right)^{d/2-\nu} \phi_-(x^\mu) + i\left(\frac{u}{u_0}\right)^{d/2+\nu} \pi^{-d/2}\frac{\Gamma(\Delta_+)}{\Gamma(\nu)} u_0^{2\nu} \int d^d y \frac{\phi_-(y)}{((x-y)^2+i\epsilon)^{\Delta_+}}\,, \tag{6.3}$$

from which we easily read off the subleading behavior. We can do a similar exercise for the retarded and advanced cases, and we find that the leading and subleading behaviors are:

$$\begin{aligned} \Phi(u,x^\mu) \simeq & \left(\frac{u}{u_0}\right)^{d/2-\nu} \phi_-(x^\mu) + 2\sin\pi\Delta_+ \left(\frac{u}{u_0}\right)^{d/2+\nu} \pi^{-d/2}\frac{\Gamma(\Delta_+)}{\Gamma(\nu)} u_0^{2\nu} \times \\ & \times \int d^d y\, \theta(\pm(x^0-y^0)) \frac{\phi_-(y)}{(-(x-y)^2_+)^{\Delta_+}}, \end{aligned} \tag{6.4}$$

where we used the Fourier transform of $(p^2 \mp i\epsilon\,\mathrm{sign}(E))^\nu$ which can be found in [31] and we have made use of the generalised function x_+^λ defined in [30] as (roughly) being x^λ when $x > 0$ and zero otherwise. We have thus explicitly seen how the subleading behavior ϕ_+ of the wavefunction is related to the leading behavior ϕ_-. The interpretation of the subleading term is derived in the next subsection.

6.3 Two-point functions

To compute two-point functions in the different prescriptions, we follow the appendix of [32] (see also [33]), and write the bulk-to-boundary propagators in momentum space as:

$$\begin{aligned} K_F(u,p^\mu) &= \frac{u^{d/2} H_\nu^{(1)}(u\sqrt{-p^2+i\epsilon})}{u_0^{d/2} H_\nu^{(1)}(u_0\sqrt{-p^2+i\epsilon})} \\ K_{R,A}(u,p^\mu) &= \theta(\pm E)\frac{u^{d/2} H_\nu^{(1)}(u\sqrt{-p^2+i\epsilon})}{u_0^{d/2} H_\nu^{(1)}(u_0\sqrt{-p^2+i\epsilon})} + \theta(\mp E)\frac{u^{d/2} H_\nu^{(2)}(u\sqrt{-p^2-i\epsilon})}{u_0^{d/2} H_\nu^{(2)}(u_0\sqrt{-p^2-i\epsilon})}\,. \end{aligned} \tag{6.5}$$

We need to get rid of contact terms, and we see that it is important not to take the limit $u_0 \to 0$ too fast, not to loose the right normalisation for the two-point function [32]. Following the appendix of [32], we obtain precisely the same boundary two-point function in momentum space as the authors of [32] did in euclidean signature, but for the Feynman two-point function we need to replace $p^{2\nu}$ by $(p^2 - i\epsilon)^\nu$ (and there is an overall factor of i). To obtain this, we crucially made use of the fact that for $u \to \infty$ the bulk-to-boundary Green function vanishes (i.e. $K_B \to 0$ for $u \to \infty$) to (automatically) get rid of a boundary term. We thus recuperate the two-point functions in [14], but in a more transparant fashion. We obtain precisely the boundary (Feynman) time-ordered two-point function. By a familiar Fourier transform, we get the two-point correlator in position space.

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By plugging in a $\delta-$function source term for the boundary field following [11, 24], we can interpret the subleading behavior of the bulk solution in (6.3) as the time-ordered expectation value of an operator, in the presence of another operator insertion up to an overall normalisation, which we claim coincides with the one found in [24]. To derive in a precise manner the normalisation constant requires more care in the regularisation scheme (see e.g [34, 35]).

We can repeat a similar computation for the retarded and advanced two-point function, but we have to be careful when defining the two-point function in terms of the action functional. After some computation, we find in position space the following action functional for the case of the retarded bulk-to-boundary propagator (where we suppress a double integration over the boundary space):

$$S[\phi_-(x), \phi_-(y)] = \phi_-(x)\phi_-(y)\theta(x^0 - y^0)u_0^{2\nu-d}\frac{\Gamma(\Delta_+)}{\Gamma(\nu)}\pi^{-d/2}2\nu\sin\pi\Delta_+ \cdot (-(x-y)^2)_+^{-\Delta_+} \,. \tag{6.6}$$

Clearly, a symmetric functional derivative with respect to the two source-terms will not define an (asymmetric) time-ordered two-point function. The trick we need is familiar from the closed time-path formalism for quantum field theory (see e.g. [36] for a review). We can formally introduce two independent sources $\phi_-^1(x)$ and $\phi_-^2(y)$ which are time-ordered with respect to each other. The retarded two-point function is then defined by the functional derivative with respect to these two sources (see [36] for more detail) and becomes:

$$\langle\theta(x^0 - y^0)O(x)O(y)\rangle = -i\theta(x^0 - y^0)u_0^{2\nu-d}\frac{\Gamma(\Delta_+)}{\Gamma(\nu)}\pi^{-d/2}2\nu\sin\pi\Delta_+ \cdot (-(x-y)^2)_+^{-\Delta_+} \,. \tag{6.7}$$

Note that $(-(x-y)^2)_+$ assures that the correlator is vanishing when $(x-y)^\mu$ is spacelike. Clearly, this formal trick takes into account the physically evident fact that the operator $O(x)$ (associated to the source $\phi_-(x)$) has the same time-ordering with respect to $O(y)$ as the source term $\phi_-(x)$ has with respect to $\phi_-(y)$. The interpretation of the subleading behavior of the bulk wave function is then in terms of the expectation value of an operator in the presence of an operator inserted at an earlier time. We can repeat a similar story for the advanced propagator.

What we basically observe is that time-ordering in the bulk and on the boundary agree in Poincaré coordinates, and that the bulk ϵ-prescription, via the radial boundary conditions, is directly reflected in the ϵ- or time-ordering prescription in the boundary field theory.

7. Conclusions

By treating carefully the Feynman, retarded and advanced bulk propagator, we noticed they could be rewritten in terms of a Minkowksi propagator and a radial Green function. Making use of that key fact, we used the ϵ-prescription associated to the causal structure of the bulk propagator to regulate the radial part of the bulk wavefunctions appropriately, which automatically lead to the vanishing of the boundary term in the two-point functions associated to the horizon. In this way we derived the boundary conditions on the bulk

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wavefuntions that are suited for each choice of propagator, thus putting the prescription used in [14] on a firm footing. We derived the bulk-to-boundary propagators from the bulk propagators with some care, and interpreteted the two-point functions and the subleading behavior of the bulk solution to the wave equation thus making the general statements in [11] more specific. We thus clarified key issues.

From our analysis it is clear that we obtain a full set of two-point functions, as described in the literature on time-dependent quantum field theory. We already briefly indicated the relevance of the closed time-path formalism to the interpretation of the two-point functions in section 6. For higher-point functions, we naturally expect to be able to derive correlators associated to different time-orderings of the operators and at that point the extensive formalism [36] of closed time-path ordering[3] will start playing an important role. (See also [37] for the relevance of the formalism at finite temperature.) We hope to return to this issue in the future.

Once we have a more complete dictionary for time-dependent AdS/CFT, we should be able to study enigmatic time-dependent processes in gravity (like the formation and evaporation of a black hole), or, perhaps more realistically for now, time-dependent processes in strongly coupled field theories at large N using gravity. To make the dictionary more complete, it would be useful to analyze what the interesting and manageable initial conditions are for the wave equation, and how they translate from bulk into boundary and vice versa. We took a step towards making that analysis well-founded.

Note added. While we were writing up our work, the paper [38] appeared on the archive, which follows a different route to try to justify the prescription of [14].

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A. Conventions

A.1 Coordinates and metrics

Our conventions are as follows. The space in which we embed AdS_{d+1} has metric:

$$ds^2 = -(dX^0)^2 + (dX^1)^2 + \cdots + (dX^d)^2 - (dX^{d+1})^2 . \tag{A.1}$$

The space AdS_{d+1} is defined by:

$$-(X^0)^2 + (X^1)^2 + \cdots + (X^d)^2 - (X^{d+1})^2 = -l^2 \tag{A.2}$$

[3]We thank Yoav Bergner and Luis Bettencourt for explaining parts of the formalism to us.

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and we put $l = 1$. We will frequently use the nomenclature AdS_{d+1} for the cover of this space. Our Poincaré coordinates we define as follows[4]:

$$\begin{aligned} X^\mu &= \frac{x^\mu}{u} \qquad \mu \in \{0, 1, \ldots, d-1\} \\ X^d &= \frac{1-u^2}{2u} - \frac{x^2}{2u} \\ X^{d+1} &= \frac{1+u^2}{2u} + \frac{x^2}{2u} \\ r &= \frac{1}{u} \\ t &= x^0 \end{aligned} \tag{A.3}$$

and the metric in these coordinates is:

$$\begin{aligned} ds^2 &= u^{-2}(dx^\mu)^2 + \frac{du^2}{u^2} \\ ds^2 &= r^2(dx^\mu)^2 + \frac{dr^2}{r^2} \,. \end{aligned} \tag{A.4}$$

The index j will run over the spatial coordinates of the Minkowski space.

A.2 Green functions in curved spaces

We discuss our conventions for Green functions in curved space which mostly coincide with those of [39].

- The Green functions for a scalar field $\phi(x)$ are defined by:

$$\begin{aligned} G^+(x,x') &= \langle 0|\phi(x)\phi(x')|0\rangle \\ G^-(x,x') &= \langle 0|\phi(x')\phi(x)|0\rangle \\ iG(x,x') &= \langle 0|[\phi(x),\phi(x')]|0\rangle \\ G^{(1)}(x,x') &= \langle 0|\{\phi(x),\phi(x')\}|0\rangle \\ G_R(x,x') &= -\theta(t-t')G(x,x')\,, \qquad G_A(x,x') = +\theta(t'-t)G(x,x') \\ iG_F(x,x') &= \langle 0|T(\phi(x)\phi(x'))|0\rangle = -\frac{i}{2}\Big(G_R(x,x') + G_A(x,x')\Big) + \frac{1}{2}G^{(1)}(x,x')\,. \end{aligned} \tag{A.5}$$

Alternatively, we have for the bulk propagators G_B:

$$\begin{aligned} G_R &= i\theta(t-t')(G^+ - G^-) \\ G_A &= -i\theta(t'-t)(G^+ - G^-) \\ -G_F &= i\theta(t-t')G^+ + i\theta(t'-t)G^-\,. \end{aligned} \tag{A.6}$$

These bulk Green functions all have the same source term (including the sign).

[4]Note that r is a logical radial coordinate, increasing as we go towards the boundary, while $u = 1/r$. We have mostly plus signature in both the embedding and the Minkowski space.

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- The differential equation satisfied by $\phi(x)$ is:

$$(\Box - m^2)\phi(x) = 0\,, \qquad \Box = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu\,. \tag{A.7}$$

We assume the static geometry with the metric

$$ds^2 = g_{00}(x^i)dt^2 + g_{ij}(x^i)dx^i dx^j\,, \qquad (i,j = 1,2,\ldots)\,. \tag{A.8}$$

- The differential equations satisfied by the Green functions are:

$$\begin{aligned}(\Box_x - m^2)G^{\pm}(x,x') &= (\Box_x - m^2)G(x,x') = (\Box_x - m^2)G^{(1)}(x,x') = 0\\ (\Box_x - m^2)G_{R,A}(x,x') &= \frac{-1}{\sqrt{-g}}\delta(x-x')\\ (\Box_x - m^2)G_F(x,x') &= \frac{+1}{\sqrt{-g}}\delta(x-x')\end{aligned} \tag{A.9}$$

where we made use of the equations:

$$\begin{aligned}\Box(AB) &= (\Box A)B + 2g^{\mu\nu}\partial_\mu A\partial_\nu B + A(\Box B)\\ \Box\theta(t) &= \frac{1}{\sqrt{-g}}\partial_0\sqrt{-g}g^{00}\delta(t) = g^{00}\delta'(t)\end{aligned} \tag{A.10}$$

and the canonical commutation relations are:

$$[\phi(x),\dot\phi(x')]_{t=t'} = \frac{-i}{g^{00}\sqrt{-g}}\delta^{(d-1)}(\vec{x}-\vec{x'})\,, \quad (\vec{x} = (x^i)). \tag{A.11}$$

A.3 Green functions in Minkowski space

Our conventions for Minkowksi space agree with the ones we specified for a general curved space. Then the Green functions G_B for Minkowski space and expectation values $G^{\pm}$ are associated to the following contours:

$$\begin{aligned}G^+(x,x';\lambda) &= \int \frac{d^dp}{(2\pi)^{(d-1)}}e^{ip.(x-x')}\theta(E)\delta(p^2+\lambda)\\ &= \int \frac{d^{d-1}p}{(2\pi)^{(d-1)}}\int_{C^+}\frac{dE}{2\pi i}\frac{1}{p^2+s^2}e^{ip\cdot(x-x')}\end{aligned} \tag{A.12}$$

where C^+ is the contour that circles $E_p = \sqrt{p_jp^j+s^2} > 0$ clockwise. The same expression is valid for G^- where C^- encircles $-E_p < 0$ counterclockwise.

For the bulk Green functions that we are interested in, and in our conventions, we obtain:

$$G^R(x,x';\lambda) = i\int \frac{d^{d-1}p}{(2\pi)^{(d-1)}}\int_{C^R}\frac{dE}{2\pi i}\frac{1}{p^2+s^2}e^{ip\cdot(x-x')} \tag{A.13}$$

where the retarded contour C^R is just above the real axis, leaving the poles underneath (and directed toward $+\infty$), while for G^A we find the same expression for the integral, but now with a contour just below the real E-axis (and oriented toward $+\infty$). For $-G_F$ we also have the same expression, but now the contour C^F dips at $-E_p < 0$ and rises at $E_p > 0$. The associated ϵ-prescriptions are easily worked out.

A.4 $G^{\pm}$ in coordinate space

We can compute the Minkowski Wightman functions if we regularize x^2 as $x^2 \to x^2 \pm i\epsilon\,\mathrm{sign}(t)$ to make the integral in the last step well-defined:

$$\begin{aligned} G^{\pm}(x) &= \int \frac{d^{d-1}p_j}{(2\pi)^{d-1}} \frac{1}{2E_p} e^{-iE_p t} e^{ip_j x^j} \\ &= \int_0^\infty \frac{dp p^{d-2}}{(2\pi)^{d-1}} \int d\Omega_{d-2} \frac{1}{2E_p} e^{-iE_p t} e^{ipx\cos\theta} \\ &= \frac{1}{(2\pi)^{d/2+1/1}} \pi x^{-d/2+3/2} \int_0^\infty dp p^{d/2-1/2} e^{-iE_p t} \frac{1}{E_p} J_{d/2-3/2}(px) \\ &= (2\pi)^{-d/2} \left(\frac{s}{\sqrt{x^2}}\right)^{d/2-1} K_{d/2-1}\left(s\sqrt{x^2 \pm i\epsilon\,\mathrm{sign}(t)}\right). \end{aligned} \tag{A.14}$$

B. Poincaré goes global

The purpose of this appendix is to compare the bulk propagators in Poincaré coordinates that we derived in section 3, to the propagators in global coordinates.

B.1 Feynman propagator

Let's concentrate on the Feynman propagator. We will be able to compute it using the technique discussed in section 3, and we can then compare it to the Feynman propagator in global coordinates. To compute the Feynman propagator, we regularize with the ϵ prescription $\frac{1}{\lambda+p^2-i\epsilon}$ in momentum space. The Fourier transform [30] is then given by:

$$\begin{aligned} iG_F &= (2\pi)^{-\frac{d}{2}} \left(\sqrt{x^2+i\epsilon}\right)^{\frac{-d+2}{2}} (rr')^{-\frac{d}{2}} \int_0^\infty \frac{d\lambda}{2} \lambda^{\frac{d-2}{4}} J_\nu(\sqrt{\lambda}u) J_\nu(\sqrt{\lambda}u') K_{\frac{d-2}{2}}\left(\sqrt{\lambda}\sqrt{x^2+i\epsilon}\right) \\ &= (2\pi)^{-\frac{d+1}{2}} e^{-i\pi\frac{d-1}{2}} ((z+i\epsilon)^2-1)^{-\frac{d-1}{4}} Q^{\frac{d-1}{2}}_{\nu-\frac{1}{2}}(z+i\epsilon) \end{aligned} \tag{B.1}$$

where $z = \sigma + 1 = \frac{1}{2}(\frac{r}{r'} + \frac{r'}{r} + rr'x^2)$ and we used that $0 < r < \infty$. We moreover made use of the formula:

$$\begin{aligned} \int_0^\infty x^{\mu+1/2} J_\nu(\beta x) K_\mu(\alpha x) J_\nu(xy)(xy)^{1/2} dx &= (2\pi)^{-1/2} \alpha^\mu \beta^{-\mu-1} y^{-\mu-1/2} e^{-(\mu+1/2)\pi i} \times \\ &\quad \times (z^2-1)^{-\mu/2-1/4} Q^{\mu+1/2}_{\nu-1/2}(z) \end{aligned} \tag{B.2}$$

which holds for

$$y > 0\,; \quad \mathrm{Re}(\alpha) > |\,\mathrm{Im}(\beta)|\,; \quad \mathrm{Re}(\nu) > -1\,; \quad \mathrm{Re}(\mu+\nu) > -1\,; \quad 2\beta y z = \alpha^2 + \beta^2 + y^2\,. \tag{B.3}$$

These conditions are satisfied once the ϵ regularisation prescription is taken into account. Now, using the connection between the Legendre function Q and the hypergeometric function [40] (for $|z| > 1$ and otherwise by analytic continuation):

$$\begin{aligned} Q^\mu_\rho(z) &= e^{i\mu\pi} 2^{-\rho-1} \sqrt{\pi} \frac{\Gamma(\rho+\mu+1)}{\Gamma(\rho+3/2)} z^{-\rho-\mu-1} (z^2-1)^{\mu/2} \times \\ &\quad \times F\left(1 + \frac{\rho}{2} + \frac{\mu}{2}, \frac{1}{2} + \frac{\rho}{2} + \frac{\mu}{2}; \rho + \frac{3}{2}; z^{-2}\right) \end{aligned} \tag{B.4}$$

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we are able to rewrite the Feynman (i.e. causal) Green function as:

$$iG_F = 2^{-\frac{d}{2}-\nu-1}\pi^{-\frac{d}{2}}\frac{\Gamma(\nu+\frac{d}{2})}{\Gamma(\nu+1)}z^{-\nu-\frac{d}{2}}F\left(\frac{d}{4}+\frac{\nu}{2}+\frac{1}{2},\frac{d}{4}+\frac{\nu}{2};\nu+1;(z+i\epsilon)^{-2}\right). \tag{B.5}$$

This matches with the Feynman Green function computed in [41] in global coordinates. We thus verified explicitly that the Feynman propagator agrees in Poincaré and global coordinates. That is as expected since it is known that the vacuum based on the concept of positive frequency modes using Poincaré time, agrees with the vacuum in global coordinates, up to some constant vacuum energy (see e.g. [25, 27]). We therefore expect the Feynman propagator to agree in both coordinate systems, and they do.

B.2 Retarded (advanced) propagator

We briefly discuss the retarded (and advanced) propagator in the Poincaré patch. Since the Wightman functions $G^{\pm}$ will be given by the prescription $x^2 \pm i\epsilon\,\mathrm{sign}\,x^0$, as in Minkowski space (see appendix A for the latter), and taking into account the definition of G_R in terms of $G^{\pm}$, we conclude that the retarded Green function in Poincaré coordinates G_R will be given by:

$$G_R = \theta(t-t')[iG(z+i\epsilon) - iG(z-i\epsilon)] \tag{B.6}$$

as in global coordinates (since the Feyman functions agree), except for the leading θ-function, which differs in Poincaré and global coordinates. Here, $G(z)$ is given by G_F in (B.5) without the $i\epsilon$ prescription. A similar expression can be obtained for G_A.

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